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SPLINE COLLOCATION APPROACH TO BOUNDARY VALUE PROBLEMS

H. D. DOCTOR, A. B. BULSARI AND N. L. KALTHIA

Department of Mathematics, S.V. Regional College of Engineering and Technology, Surat-395007, India.

SUMMARY

A well-known Stokes problem is discussed by a cubic spline collocation method. Two consecutive cubic splines are obtained for the problem. The results by this method are compared with those of an orthogonal collocation method. The selection of the length of the subintervals of the range of the boundary value problem is also justified. The results obtained by these two methods are compared with the analytic solution. The methods involve simple algebra, and hence the calculations do not require the help of a computer. Necessary error analysis has been carried out.

KEY WORDS Approximation Cubic Splines Orthogonal Collocation

INTRODUCTION

Numerical methods such as finite difference, Runge–Kutta, Milne and Taylor's series expansion for solving linear and non-linear differential equations sometimes require much computational work and time in the case of complicated equations. Bickley[1] has suggested the method of cubic splines to solve a linear two point boundary value problem; this method reduces the boundary value problem (BVP) to the problem of solving a set of linear algebraic equations. The coefficient matrix of these equations takes the form of an upper triangular Hessenberg matrix. The range of the BVP is divided into a number of equal subintervals. The results obtained are quite encouraging. Fyfe[2] discussed the error estimate to improve the results from which one can say that reducing the size of the subintervals is extremely helpful to obtain more accuracy.

Here we shall solve a well-known Stokes problem by the cubic spline method and compare the results so obtained with those of the orthogonal collocation obtained by Bulsari.[3]

METHOD OF CUBIC SPLINE COLLOCATION

Let $f(x)$ be a function with continuous derivatives in a range $[a, b]$. Divide the range into n equal intervals with the points $a = x_0 < x_1 < \ldots < x_n = b$. Then an interpolant $S(x)$ to $f(x)$ is a cubic spline where

(i) $S(x)$ is a cubic polynomial in each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, 2 \ldots n$

(ii) $S(x_i) = f(x_i)$, $i = 0, 1, 2 \ldots n$

(iii) $S'(x), S''(x)$ are continuous in $[a, b]$.

The points $x_0, x_1, x_2, \ldots, x_n$ are called the knots of the cubic spline $S(x)$.

The differential equation is now solved by the method of cubic splines suggested by Bickley[1] to solve a linear two point BVP (of second order)

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

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with boundary conditions

$$\left.\begin{aligned}\alpha_0 y + \beta_0 y' = \gamma_0, \quad \text{at} \quad x = a\\ \alpha_n y - \beta_n y' = \gamma_n, \quad \text{at} \quad x = b\end{aligned}\right\} \tag{2}$$

The function

$$S(x) = a_0 + b_0(x - x_0) + \tfrac{1}{2}c_0(x - x_0)^2 + \tfrac{1}{6}\sum_{k}^{n-1} d_k(x - x_k)_+^3; \qquad a \le x \le b \tag{3}$$

is a cubic spline where

$$\begin{aligned} x_+ &= x, \quad \text{if} \quad x \ge o\\ &= 0, \quad \text{if} \quad x < 0\end{aligned}$$

Substituting the expressions for $S(x)$, $S'(x)$, $S''(x)$ from equation (3) in equation (1) we get

$$\begin{aligned} c_0 + \sum_{u=1}^{n-1} d_k(x_i - x_k)_+ + p_i\left[b_0 + c_0(x_i - x_0)_+ + \tfrac{1}{2}\sum_{k=1}^{n-1} d_k(x_i - x_k)_+^2\right]\\ + q_i\left[a_0 + b_0(x_i - x_0)_+ + \tfrac{1}{2}c_0(x_i - x_0)_+^2 + \tfrac{1}{6}\sum_{k=1}^{n-1} d_k(x_i - x_k)_+^3\right] = r_i \end{aligned} \tag{4}$$

where $p_i = p(x_i)$, $q_i = q(x_i)$, $r_i = r(x_i)$.

The boundary conditions become

$$\alpha_0 a_0 + \beta_0 b_0 = \gamma_0 \tag{5}$$

$$\begin{aligned} \alpha_n\left[a_0 + b_0(x_n - x_0) + \tfrac{1}{2}c_0(x_n - x_0)^2 + \tfrac{1}{6}\sum_{k=1}^{n-1} d_k(x_n - x_k)_+^3\right]\\ - \beta_n\left[b_0 + c_0(x_n - x_0) + \tfrac{1}{2}\sum_{k=1}^{n-1} d_k(x_n - x_k)_+^2\right] = \gamma_n \end{aligned} \tag{6}$$

Taking $i = n(-1)0$ in equations (4), (5) and (6), $(n+3)$ linear equations in $(n+3)$ unknowns $a_0, b_0, c_0, d_0, d_1, \ldots, d_{n-1}$ are obtained. The coefficient matrix of these $(n+3)$ equations form an upper Hessenberg matrix system whose solution then determines (3) completely.

Let the cubic spline obtained above be denoted by $S^0(x)$. A better approximation $S^1(x)$ can be derived from the analysis given by Fyfe.[2]

As $S(x)$ and $S^0(x)$ are cubic splines, the error $\varepsilon(x)$ defined by

$$\varepsilon(x) = S(x) - S^0(x) \tag{7}$$

is also a cubic spline.

Here $S^0(x)$ satisfies the relation

$$S_i^{(0)''} + p_i S_i^{(0)'} + q_i S_i^{(0)} = r_i \tag{8}$$

The following equations are derived from the error analysis

$$\alpha_0 \varepsilon_0 + \beta_0 \varepsilon_0' = 0 \tag{9}$$

$$\alpha_n \varepsilon_n - \beta_n \varepsilon_n' = 0 \tag{10}$$

and the cubic spline $\varepsilon(x)$ at the internal points is given by

$$\varepsilon_i''+p_i\varepsilon_i'+q_i\varepsilon_i=\begin{cases}-\dfrac{h}{12}(2d_1-d_0) & i=0\\ -\dfrac{h}{12}d_i & i=1(1)n-1\\ -\dfrac{h}{12}(2d_{n-1}-d_{n-2}), & i=n\end{cases}\tag{11}$$

The equations (9), (10) and (11) are the same as the equations (4), (5) (6), differing only on the right hand side. However the coefficient matrix remains the same as that for (4), (5) and (6). The solution $\varepsilon(x)$ of these equations then modifies $S(x)$ by $S^1(x)=S^0(x)+\varepsilon(x)$.

CUBIC SPLINE METHOD APPLIED TO STOKES PROBLEM

Let us consider a well-known Stokes problem[4] dealing with the flow over an accelerated flat plate in a horiziontal plane.

The equation of motion is described by

$$f''(\eta)+2\eta f'(\eta)=0\tag{12}$$

with

$$\eta=\frac{y}{2\sqrt{(\gamma t)}},\qquad u=U_0f(\eta)$$

The boundary conditions are

$$\left.\begin{aligned}f&=1, &&\text{at}\quad \eta=0\\ f&=0, &&\text{as}\quad \eta\to\infty\end{aligned}\right\}\tag{13}$$

The analytic solution of the problem shows that the velocity becomes zero as η approaches 2. Hence we restrict the range $(0,\infty)$ to $[0,2]$ in order to apply the above method. The solution is obtained for $n=4$ and $n=8$. However, in what follows the analysis is given for $n=8$ only.

The boundary conditions (13) determine

$$\left.\begin{aligned}&a_0=1, \quad \beta_0=0, \quad \gamma_0=1\\ &\alpha_8=1, \quad \beta_8=0, \quad \gamma_8=0{\cdot}004678\\ &\text{From equation (5)}, \quad a_0=1\end{aligned}\right\}\tag{14}$$

Equation (4) with $i=7(-1)0$ and equations (6) and (14) give a system of ten linear equations whose solution determines the unknowns $a_0, b_0, c_0, d_0, d_1, \ldots, d_1, \ldots, d_7$. The equations are solved by Crout's algorithm and the cubic spline $S^0(\eta)$ is given by

$$\begin{aligned}S^0(\eta)=1-1{\cdot}118997\eta+\tfrac{1}{6}[&2{\cdot}106347\eta^3-0{\cdot}702117(\eta-0{\cdot}25)_+^3\\ &-1{\cdot}034692(\eta-0{\cdot}5)_+^3-0{\cdot}886888(\eta-0{\cdot}75)_+^3\\ &-0{\cdot}443441(\eta-1)_+^3+0{\cdot}000002(\eta-1{\cdot}25)_+^3\\ &+0{\cdot}250642(\eta-1{\cdot}5)_+^3+0{\cdot}292410(\eta-1{\cdot}75)_+^3]\end{aligned}\tag{15}$$

The cubic spline $S^0(\eta)$ is tabulated for different values of η in Table II. From the numerical

Table I. Velocity profiles obtained through orthogonal collecation[3]

η	Exact solution	First approximation	Second approximation	Third approximation
0·00	1·000000	1·000000	1·000000	1·000000
0·10	0·887637	0·871515	0·883239	0·885865
0·20	0·777537	0·761401	0·760445	0·774233
0·30	0·671573	0·666647	0·640758	0·667914
0·40	0·571608	0·584952	0·529479	0·569002
0·50	0·479500	0·514341	0·429347	0·478835
0·60	0·396144	0·453158	0·341418	0·398076
0·70	0·322199	0·400016	0·265689	0·326849
0·80	0·257899	0·353747	0·201539	0·264881
0·90	0·203092	0·313368	0·147934	0·211624
1·00	0·157299	0·278049	0·103771	0·166366
1·10	0·119795	0·247987	0·067849	0·128304
1·20	0·089636	0·219888	0·039018	0·096611
1·30	0·065992	0·195946	0·016210	0·070475
1·40	0·047715	0·174829	0·001542	0·049227
1·50	0·033895	0·156169	−0·015092	0·032096
1.60	0.023625	0.139651	−0.025186	0·018396
1·70	0·016210	0·125006	−0·032403	−0·007613
1·80	0·010909	0·111999	−0·037468	−0·000746
1·90	0·007210	0·100433	−0·040672	−0·007106
2·00	0·004678	0·096131	−0·042414	−0·001183

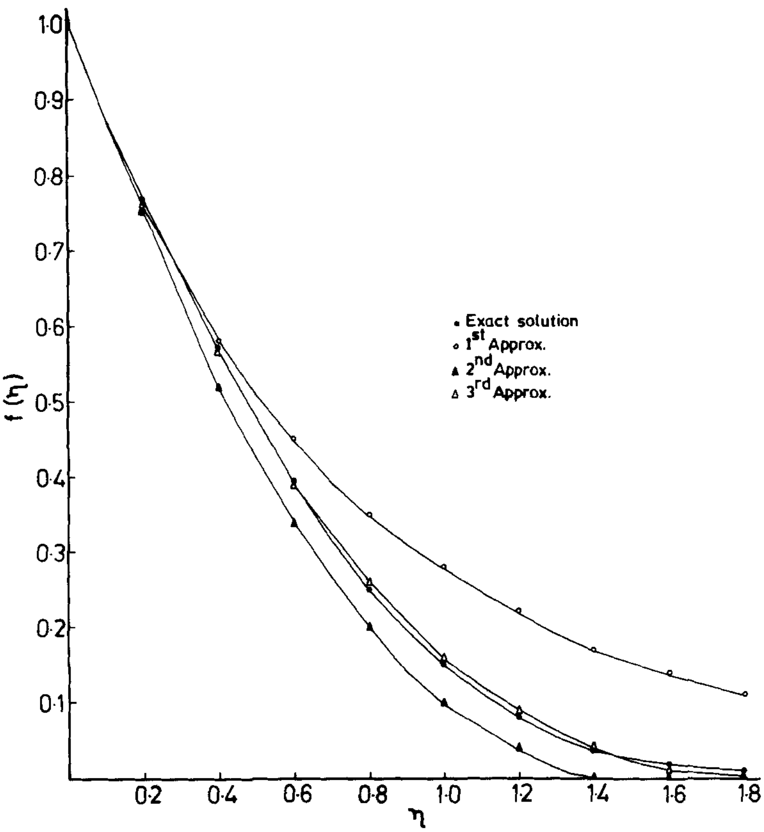

Figure 1

Table II. Velocity profiles obtained through spline collocation

η	Analytical solution	$S(\eta)$ $n=4$	$S(\eta)$ $n=8$	$\lvert f(\eta)-S(\eta)\rvert \times 10^{-4}$ $n=8$
0·000	1·000000	1·000000	1·000000	0·00
0·10	0·887537	0·885050	0·887084	4·53
0·20	0·777537	0·773745	0·776783	7.54
0·30	0·671573	0·667524	0·670970	6·03
0·40	0·571608	0·567845	0·571300	3·08
0·50	0·479500	0·476161	0·479306	2·94
0·60	0·396144	0·393660	0·395929	2·75
0·70	0·322199	0·320467	0·321982	2·17
0·80	0·257899	0·256446	0·257680	2·19
0·90	0·203092	0·201457	0·202902	1·90
1·00	0·157299	0·155361	0·157093	2·06
1·10	0·119795	0·117862	0·116092	37·03
1·20	0·059636	0·088027	0·089526	1·10
1·30	0·065992	0·064769	0·065850	1·42
1·40	0·047715	0·046996	0·047603	1·12
1·50	0·033895	0·033620	0·033825	0·70
1·60	0·023652	0·023632	0·023599	0·53
1·70	0·016210	0·016347	0·016188	0·22
1·80	0·010909	0·011163	0·010909	0·00
1·90	0·007210	0·007476	0·007212	0·02
2·00	0·004678	0·004683	0·004683	0·05

point of view the results are very close to the analytic solution (coinciding up to three decimal places) except at a very few points. However a better approximation $S^1(\eta)$ is obtained using the equations (9), (10) and (11) and $S^1(\eta)=S^0(\eta)+\varepsilon(\eta)$. The new $S^1(\eta)$ is obtained as follows

$$\begin{aligned} S^1(\eta) = {} & 1+1{\cdot}135958\eta+0{\cdot}073136\eta^2+\tfrac{1}{6}[1{\cdot}883589\eta^3 \\ & -0{\cdot}460207(\eta-0{\cdot}25)_+^3-1{\cdot}091161(\eta-0{\cdot}5)_+^3 \\ & -0{\cdot}965727(\eta-0{\cdot}75)_+^3-0{\cdot}421182(\eta-1)_+^3 \\ & +0{\cdot}034081(\eta-1{\cdot}25)_+^3+0{\cdot}269392(\eta-1{\cdot}5)_+^3 \\ & +0{\cdot}284427(\eta-1{\cdot}75)_+^3] \end{aligned} \tag{16}$$

The cubic spline $S^1(\eta)$ is obviously much closer to $f(\eta)$. The velocity profiles obtained through this equation are compared with the analytic one in Figure 2. The results are extremely satisfactory.

SELECTION OF THE LENGTH OF THE SUBINTERVALS

The desired accuracy is of 0·001 units for which the length h of the subinterval should be such that $b-a/h$ is a multiple of 4. Here $h_1=0{\cdot}5$, $h_2=0{\cdot}25$ are selected to obtain the accuracy. The test for h, suggested by Fyfe[2] is followed and fulfilled in this problem.

Setting $\varepsilon=0{\cdot}0001$, $\xi_k=0{\cdot}001$.

$$\phi=\frac{h^4}{384}\max_{i=1(1)n-1}|d_i|<\varepsilon$$

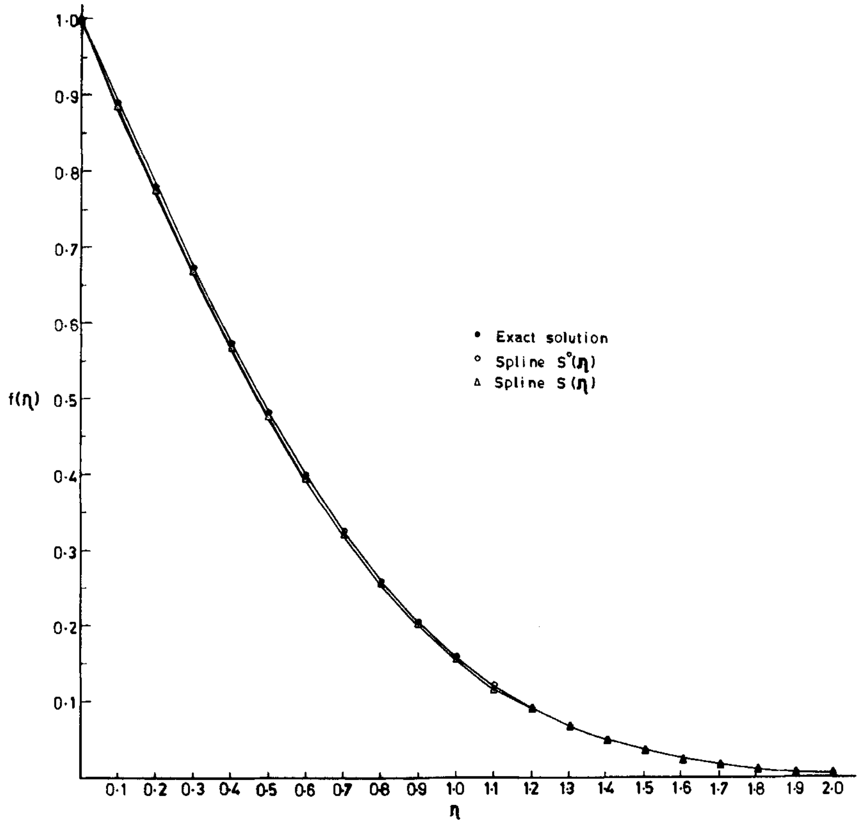

Figure 2

Here $\phi = 0{\cdot}000044 < \varepsilon$ is satisfied for $h = 0{\cdot}25 = (h_2)$. Also

$$\delta_i = \left|\frac{h_2^4}{h_1^4 - h_2^4}\right| \, \left| y_i(h_1) - y_i(h_2) \right|$$

where i denotes the common knots for both h_1 and h_2.

The maximum absolute value of δ_i is $0{\cdot}000231$ which is less than $\varepsilon_k = 0{\cdot}001$, i.e. $\delta = \max\limits_{i=1(1)n-1} \delta_i < \varepsilon_k$ is satisfied. Hence the desired accuracy is maintained. More accuracy requires higher approximation following the same procedure.

CONCLUSION

A well-known Stokes problem for the study of flow of a fluid near a suddenly accelerated plate is solved by an approximate method, i.e. cubic spline collocation. The comparison of the extremely satisfying results obtained through this method with the analytic solution and Bulsari's[3] results by orthogonal collocation induces its applicability to other fluid mechanics problems which gives rise to linear differential equations whose closed form solutions are not available. In contrast to many computerized (completely) methods such as finite element methods or finite difference methods, this method can be put through by simple desk calculations and thus saves a lot of computer time.

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